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Upstream boundary-layer separation in stratified flow

By JOHN W. MILES
Institute of Geophysics and Planetary Physics,† University of California,
La Jolla, California 92037

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Stratified, inviscid channel flow over a thin barrier or into an abrupt contraction is considered on the hypotheses that the upstream dynamic pressure and density gradient are constant (Long's model) for those parametric régimes in which the hypotheses are tenable for finite-amplitude disturbances, namely $k < 2$ for the barrier and $k < 1$ for the contraction, where $k = NH/\pi U$ is an inverse Froude number based on the Väisälä frequency N, the channel height H, and the upstream velocity U. Reverse flow in the neighbourhood of the forward stagnation point, which implies the formation of an upstream separation bubble, is found for certain critical ranges of k. The maximum barrier height for which the dominant lee-wave mode can exist without reversed flow either upstream or downstream of the barrier is $0{\cdot}34H$. The limiting case of a half space is considered briefly, and forward separation is found for $\kappa \equiv Nh/U > \kappa_s$, where $\kappa_s = 2{\cdot}05$ for a thin barrier and $1{\cdot}8$ for a semi-circular barrier. The corresponding values for reverse flow in the lee-wave field are $\kappa_c = 1{\cdot}73$ and $1{\cdot}3$, respectively.

1. Introduction

The phenomenon of upstream blocking in stratified flow, as observed by Long (1955) and Debler (1959), involves the formation of a stagnation zone (or zones) upstream of a barrier or contraction in a channel flow. Long's observations of flow over barriers suggest that blocking may be triggered by local flow reversals and density inversions, with the consequent formation of turbulent eddies, in the lee-wave field. Debler's observations of flow into a line sink (which may be regarded as the limiting case of a contraction) suggest that blocking is initiated by boundary-layer separation at the forward stagnation point. Maxworthy's (1970) more detailed observations of rotating flow past a sphere show that an upstream separation bubble definitely forms for a sufficiently large value of, and grows with, the inverse Rossby number (the analogue of the inverse Froude number, k below), although its ultimate length and shape appear to be controlled by viscous effects. It is, of course, possible that the formation of turbulent eddies in the downstream flow and forward separation are not independent, but it nevertheless appears worthwhile to show that an inviscid model does predict forward separation and to obtain explicit results for some simple configurations. [Miles (1968, 1971) develops theoretical criteria for both lee-wave instability

† Also Department of Aerospace and Mechanical Engineering Sciences.

and stagnation-point separation in rotating flow past a sphere and finds that the two critical Rossby numbers are roughly equal; however, this equality appears to be essentially coincidental, and it seems unlikely that the critical conditions for the two separate phenomena are simply related.]

We consider here stratified flow over a thin barrier or into a contraction on the basis of Long's (1953) model, in which the dynamic pressure and the vertical gradient of the density are constant (at a sufficiently large distance) upstream of the discontinuity. Let H be the height of the channel, h the height of either the thin barrier or the step in the bottom of the channel (see figure 1), $U(y)$ the speed of the undisturbed flow, and $N(y)$ the intrinsic (Väisälä) frequency in the undisturbed flow. Choosing H/π as a characteristic length, we construct the characteristic parameters

$$d = \pi h/H \equiv \pi(1-c), \tag{1.1}$$

$$k = NH/\pi U, \quad \text{and} \quad \kappa = kd \equiv Nh/U, \tag{1.2a, b}$$

where c is the contraction ratio and k is, by hypothesis, independent of y, the dimensionless elevation above the bottom of the channel. The hypothesis of *no upstream influence* implicit in Long's model holds for the barrier if $k < 2$ and for the contraction if $k < 1$. Disturbances of order κ^2 appear upstream of the barrier if $k > 2$ in consequence of second-order interactions among the internal waves (McIntyre 1971). Disturbances of order κ appear upstream of the contraction if $k > 1$ in consequence of the source-like effect of the step (Wong & Kao 1970).†

We follow a previous analysis (Miles 1968*a*) of the lee-wave problem for a thin barrier and refer to sections and equations therein by the prefix I. Proceeding as in I, we express the Cartesian components of the particle velocity in terms of the dimensionless displacement of a streamline, $\delta(x,y)$:

$$\{u, v\} = U(y-\delta)\{1-\delta_y, \delta_x\}. \tag{1.3}$$

The resulting boundary-value problem then is described by the Helmholtz equation,

$$\nabla^2\delta + k^2\delta = 0, \tag{1.4}$$

the boundary conditions (on the hypothesis of unseparated flow)

$$\delta(x, y_0) = y_0, \quad \delta(x, \pi) = 0, \tag{1.5a, b}$$

$$\delta(0, y) = y \quad (0 \leqslant y \leqslant d), \tag{1.5c}$$

and the requirements that no waves appear in the upstream flow ($x \to -\infty$) and that the singularity at $x = 0$ and $y = d$ be physically acceptable (see discussion in I). The lower boundary for $x \neq 0$ is specified by either

$$y_0 = 0 \qquad \text{(figure 1}(a)), \tag{1.6a}$$

or

$$y_0 = dH(x) \quad \text{(figure 1}(b)), \tag{1.6b}$$

where $H(x)$ is Heaviside's step function.

† Jones (1970) gives a solution for an expansion ($c < 0$) in which he invokes the hypothesis of no upstream influence for $k > 1$.

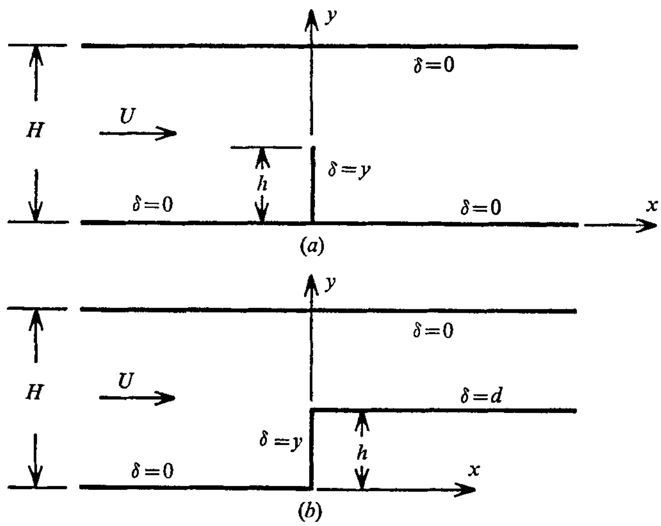

FIGURE 1. (*a*) Thin barrier in channel. (*b*) Contraction of channel.

2. Thin barrier

The solution of this boundary-value problem for the thin barrier (figure 1 (*a*)) has the form (I, §3)

$$\delta(x,y) = \left\{-2H(x)\sum_{1}^{K}\sin(\alpha_n x) + \sum_{K+1}^{\infty}\exp(-\alpha_n|x|)\right\}(G_n/\alpha_n)\sin ny, \qquad (2.1)$$

where $$\alpha_n = |n^2-k^2|^{\frac{1}{2}} \quad (K < k < K+1), \qquad (2.2)$$

K is the integral part of k, and the G_n are the Fourier coefficients of the equivalent-vortex-sheet distribution,

$$g(y) \equiv \tfrac{1}{2}\{\delta_x(0-,y)-\delta_x(0+,y)\} \qquad (2.3a)$$

$$= \sum_{1}^{\infty} G_n \sin ny, \qquad (2.3b)$$

wherein $\delta_x(0\mp,y) \equiv \partial\delta/\partial x$ at $x = 0\mp$. The dimensionless particle velocity (in the positive direction of flow) on the barrier is given by

$$q(y) = \pm\,\delta_x(0\mp,y) \qquad (2.4a)$$

$$= g(y) + \operatorname{sgn} x \sum_{1}^{K} G_n \sin ny. \qquad (2.4b)$$

The G_n are determined by the requirement that $\delta(0,y)$ be continuous in $d < y < \pi$ and are given either by the variational approximation of I, §4 or by the perturbation solution of I, §5.

Proceeding on the basis of the latter solution, we obtain

$$q(y) = q_0(y)\,[1+2a\epsilon_1\{1-\epsilon_1+\epsilon_1(1-a)^2\}^{-1}(a+\cos y)+O(\epsilon_2)] \quad (0 \leqslant k < 1) \qquad (2.5a)$$

$$= q_0(y)\,[1+2a(1-a)^{-2}(a+\cos y)+2a(a+2)\,\epsilon_2\{1-\epsilon_2+\epsilon_2(1-a)^4\}^{-1}$$
$$\times(2a+a^2+4a\cos y+\cos 2y)+O(\epsilon_3)] \quad (1 < k < 2), \qquad (2.5b)$$

where $$q_0(y) = (1-\cos y)^{\frac{1}{2}}(\cos y - \cos d)^{-\frac{1}{2}} \tag{2.6}$$
is the solution for potential flow ($k = 0$) over the barrier,
$$a = \sin^2 \tfrac{1}{2}d, \tag{2.7}$$
and $$\epsilon_n = 1-\{1-(k/n)^2\}^{-\frac{1}{2}}. \tag{2.8}$$
Letting $y \downarrow 0$, we obtain
$$q_s' = \tfrac{1}{2}a^{-\frac{1}{2}}[1+2a(1+a)\,\epsilon_1\{1-\epsilon_1+\epsilon_1(1-a)^2\}^{-1}+O(\epsilon_2)] \quad (0 \leqslant k < 1) \tag{2.9a}$$
$$= \tfrac{1}{2}a^{-\frac{1}{2}}[1+2a(1+a^{\frac{1}{2}})^{-2}+2a(a+2)\,(1+6a-4a^{\frac{3}{2}}+a^2)\,\epsilon_2\{1-\epsilon_2+\epsilon_2(1-a)^4\}^{-1} + O(\epsilon_3)] \quad (1 < k < 2). \tag{2.9b}$$
The ratio of this stagnation-point velocity gradient to its value in potential flow, $\frac{1}{2}\csc\frac{1}{2}d$, is plotted in figure 2. Reversed flow, accompanied by an upstream separation bubble, is implied by $q_s' < 0$ and occurs for $k_n(d) < k < n+1$ ($n = 0, 1$); k_0 and k_1 are plotted in figure 3;† κ_0 and κ_1 are plotted in figure 4. We remark that k_0 and k_1 are monotonically decreasing functions of d; on the other hand, κ_0 is a monotonically increasing function of d, whilst κ_1 appears to oscillate about the asymptote 1·82 ($|\kappa_1 - 1{\cdot}82| < 0{\cdot}01$ for $h/H > 0{\cdot}36$).

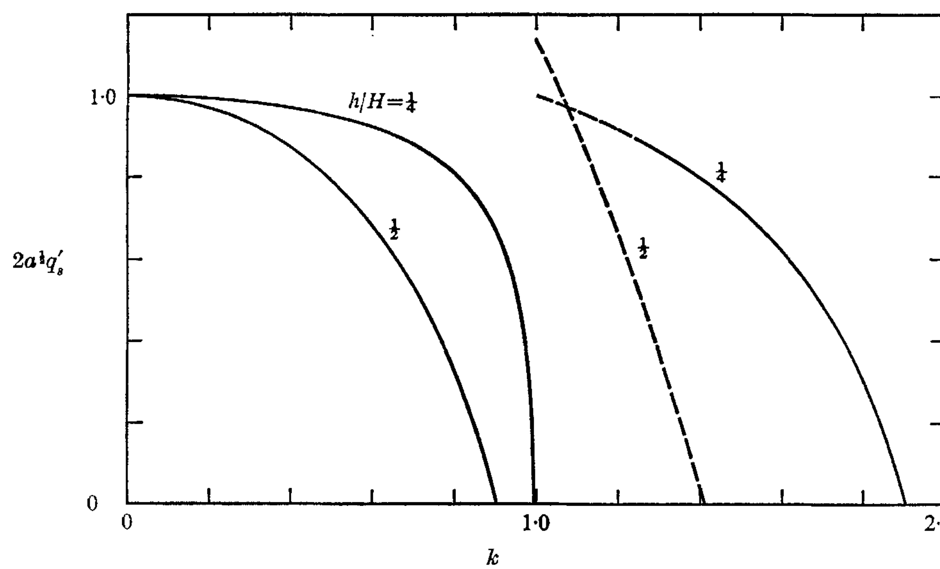

FIGURE 2. The ratio of the (upstream) stagnation-point velocity gradient, as given by (2.9), to the corresponding limit for potential flow. The dashed portions of the curves are in the parametric domain of lee-wave instability (see figure 3).

Stagnation points ($1-\delta_y = \delta_x = 0$) appear in the lee-wave field for $1 < k < k_c$ and are embedded in regions of local density inversion and reversed flow that appear to imply at least local instability of the lee waves (Long 1955). The locus $k = k_c$, at which stagnation points appear on the upper boundary, is plotted in

† The notation in §2 of the present paper differs from that of I in the definition of k_n.

figure 3 (this locus is determined by I (3.13a), which is based on the lee-wave field alone and neglects the effects of the trapped modes). It intersects $k = k_1$ at $k = 1{\cdot}66$ and $h = 0{\cdot}34H$, thereby determining the maximum barrier height for which the dominant lee-wave mode can exist without closed streamlines somewhere in the flow.

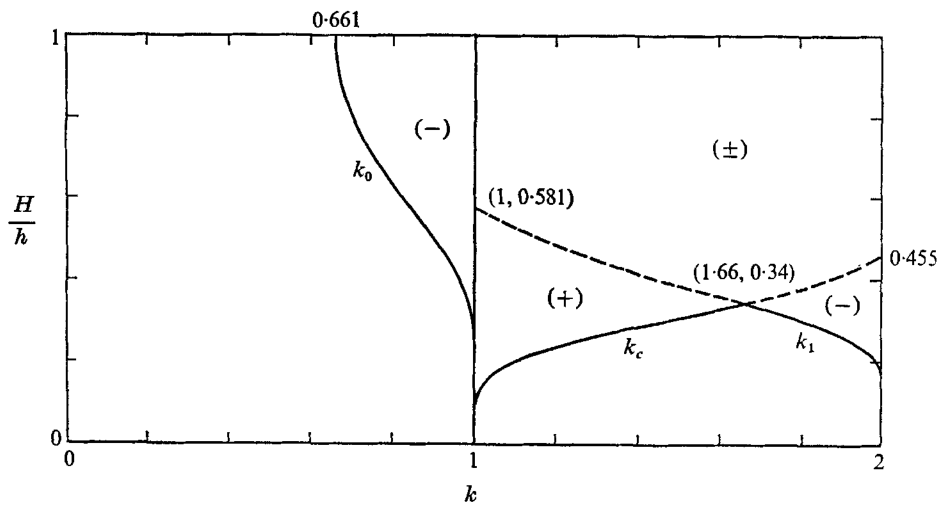

FIGURE 3. The parametric domains for the thin barrier. Flow without either lee waves or closed streamlines is possible for $0 \leqslant k < k_0$. Flow with the dominant lee-wave mode and without closed streamlines is possible for $k_c < k < k_1$. Upstream separation occurs in the ($-$) domains. Lee-wave instability occurs in the ($+$) domains.

Both upstream separation and downstream instability would appear to occur if both $k_1 < k < 2$ and $1 < k < k_c$ [the ($\pm$) domain in figure 3], but also may occur if either $k_1 < k < 2$ or $1 < k < k_c$, since either upstream separation or the existence of closed streamlines in the dowstream flow implies the failure of the hypothesis that all streamlines originate in a uniform flow at $x = -\infty$ (e.g. instability of the downstream flow could alter the upstream flow and initiate upstream separation in $k < k_1$). The structure of the solution implies that the parametric domains for the existence of the higher ($n \geqslant 2$) lee-wave modes without closed streamlines shrink rapidly with increasing n (e.g. the results of I imply lee-wave instability for $h/H > 0{\cdot}18$ at $k = 2{\cdot}5$); however, the hypothesis of no upstream influence holds only to first order (in κ) for $k > 2$, whereas the formation of closed streamlines is a finite-amplitude effect.

3. Contraction

Turning now to the contraction, we impose the *a priori* restriction $k < 1$, pose the representations

$$\delta_x(x, y) \to \sum_1^\infty A_n \sin ny \quad (x \uparrow 0,\ 0 \leqslant y \leqslant \pi) \tag{3.1a}$$

$$\to \sum_1^\infty B_n \sin nz \quad (x \downarrow 0,\ 0 \leqslant z \leqslant \pi), \tag{3.1b}$$

where
$$z = \pi(y-d)/(\pi-d) \equiv (y-d)/c, \tag{3.2}$$
and develop the solution of (1.4) and (1.5a, b) in the form
$$\delta(x,y) = \sum_1^\infty \alpha_n^{-1} A_n \exp(\alpha_n x) \sin ny \quad (x<0) \tag{3.3a}$$
$$= \delta_0(y) - \sum_1^\infty \beta_n^{-1} B_n \exp(-\beta_n x) \sin nz \quad (x>0), \tag{3.3b}$$
where α_n is given by (2.2),
$$\beta_n = [(n/c)^2 - k^2]^{\frac{1}{2}}, \tag{3.4}$$
and
$$\delta_0(y) = d \sin\{k(\pi-y)\}/\sin\{k(\pi-d)\} \tag{3.5a}$$
$$= d \sin\{kc(\pi-z)\}/\sin(kc\pi). \tag{3.5b}$$

[We note that $y - \delta_0(y) \to z$ as $k \downarrow 0$, corresponding to a uniform flow, but that the flow as $x \to \infty$ is not uniform for $k > 0$.]

Requiring $\delta(0,y)$ to be continuous in $d \leqslant y \leqslant \pi$ and invoking (1.5c) in $0 \leqslant y \leqslant d$, we obtain
$$\sum_1^\infty \alpha_n^{-1} A_n \sin ny = \delta_0(y) - \sum_1^\infty B_n \beta_n^{-1} \sin nz \quad (d \leqslant y \leqslant \pi) \tag{3.6a}$$
$$= y \quad (0 \leqslant y \leqslant d). \tag{3.6b}$$

It then remains to determine the A_n and B_n from (3.6) and the identity between (3.1a, b) in $d < y < \pi$,
$$\sum_1^\infty A_n \sin ny = \sum_1^\infty B_n \sin nz \quad (d < y < \pi). \tag{3.7}$$

Multiplying (3.7) through by $(2/\pi)\sin mz$ and integrating from $z = 0$ to $z = \pi$, we obtain
$$B_m = \sum_1^\infty I_{mn} A_n \quad (m = 1, 2, \ldots, \infty), \tag{3.8}$$
where
$$I_{mn} = (2/\pi)\int_0^\pi \sin mz \sin ny \, dz \tag{3.9a}$$
$$= (2m/\pi)(m^2 - n^2c^2)^{-1} \sin nd. \tag{3.9b}$$

Substituting (3.8) into (3.6a), multiplying (3.6a, b) through by $(2/\pi)\sin my$, and integrating from $y = 0$ to $y = \pi$, we obtain the infinite set of simultaneous equations
$$\alpha_m^{-1} A_m + c \sum_{l=1}^\infty \sum_{n=1}^\infty \beta_l^{-1} I_{lm} I_{ln} A_n = D_m \quad (m = 1, 2, \ldots, \infty), \tag{3.10}$$
where
$$D_m = (2/\pi)\left\{\int_0^d y + \int_d^\pi \delta_0(y)\right\} \sin my \, dy \tag{3.11a}$$
$$= (2/\pi)\,[\{m^{-2} + kd(m^2-k^2)^{-1} \cot k(\pi-d)\} \sin md + k^2 d m^{-1}(m^2-k^2)^{-1} \cos md]. \tag{3.11b}$$

A first approximation to the solution of (3.10) is given by (higher approximations may be obtained by integration)

$$A_n^{(1)} = \alpha_n D_n. \tag{3.12}$$

We test this approximation by calculating $q(y)$ in the limit of potential flow, $k \downarrow 0$. Invoking this limit in (3.11) and (3.12) and substituting the result for $A_n^{(1)}$ into (3.1 *a*), we obtain

$$q^{(1)}(y) = (2/\pi) \sum_1^\infty (nc)^{-1} \sin nd \sin ny \quad (k = 0) \tag{3.13a}$$

$$= (\pi c)^{-1} \log \{\sin \tfrac{1}{2}(d+y)/\sin \tfrac{1}{2}|d-y|\} \tag{3.13b}$$

$$\rightarrow (\pi c)^{-1} \tan(\tfrac{1}{2}\pi c)\, y \quad (y \downarrow 0). \tag{3.13c}$$

The exact result for potential flow is given by (Milne-Thomson 1960, §10.7)

$$\tfrac{1}{2}y = \tan^{-1} q - c \tan^{-1} cq \quad (0 \leqslant y < d) \tag{3.14a}$$

$$\rightarrow (1-c^2)q \quad (y \downarrow 0). \tag{3.14b}$$

Comparing $q_s' = (\pi c)^{-1} \tan(\frac{1}{2}\pi c)$, as given by (3.13 *c*), with the exact result $q_s' = \frac{1}{2}(1-c^2)^{-1}$, as given by (3.14 *b*), we find that the error is less than 4·5 % for $c < \frac{1}{2}$ and that the limiting ratio of the two results as $c \uparrow 1$ is $8/\pi^2$. The comparison near $y = d$, where (3.13 *b*) yields a logarithmic singularity in contrast to the $|d-y|^{-\frac{1}{2}}$ singularity implied by (3.14 *a*), is, of course, less favourable, but this is of no great import for the present investigation.

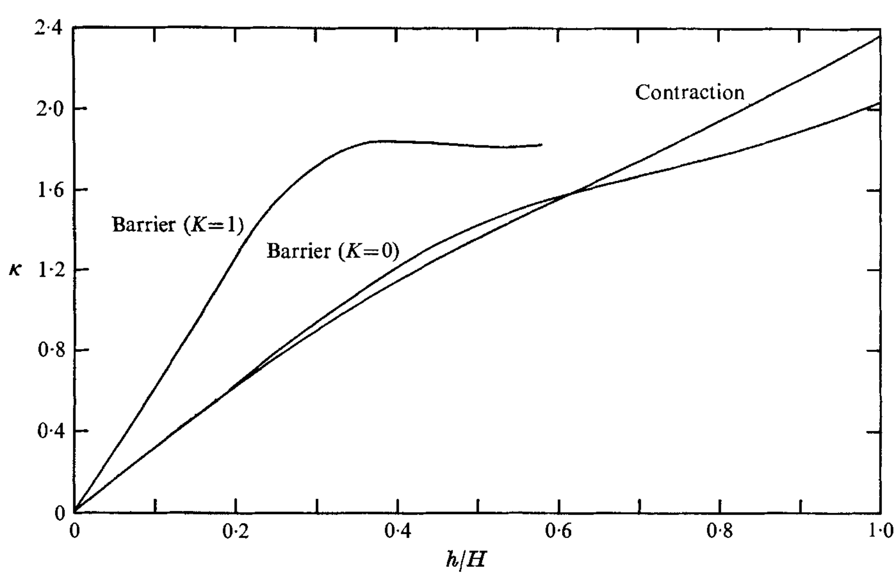

FIGURE 4. The separation parameters κ_0 and κ_1 for the barrier and κ_0 for the contraction. K is the integral part of k; $\kappa = kd$.

Substituting (3.12) into (3.1 *a*), differentiating with respect to y, and setting $y = 0$, we obtain

$$q_s'^{(1)} = \sum_1^\infty n\alpha_n D_n. \tag{3.15}$$

This series is formally divergent (in consequence of the non-uniform convergence of the Fourier series for $q(y)$ as $y \downarrow 0$) but may be transformed to a convergent (like $1/n^2$) series by separating out the asymptotic approximations to the summands in the series for $q(y)$ as $n/k \to \infty$ and summing these terms prior to differentiation with respect to y. Carrying out this procedure, we obtain

$$\pi q_s'^{(1)} = \{1 + kd \cot k(\pi - d)\} \cot \tfrac{1}{2}d + 2k^2 d \log (\tfrac{1}{2} \csc \tfrac{1}{2}d) + 2\sum_1^\infty \{(n/\alpha_n) - 1\}$$
$$\times [\{kd \cot k(\pi - d) - (\alpha_n/n)\} \sin nd + (k^2 d/n) \cos nd]. \quad (3.16)$$

The variation of q_s' with k in $0 < k < k_0$ is qualitatively similar to that for the thin barrier (figure 2). The parameter k_0, defined as in §2, such that $q_s' = 0$ at $k = k_0$, is plotted in figure 5. The parameter $\kappa_0 = k_0 d$, which is a monotonically increasing function of d, is plotted in figure 4.

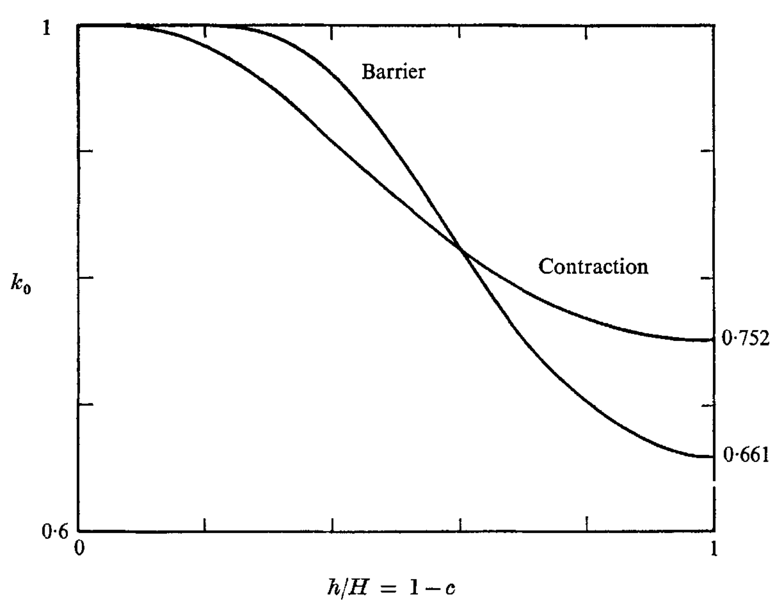

FIGURE 5. The separation parameter k_0 for the contraction compared with that for the barrier. Forward separation occurs for $k_0 < k < 1$.

4. Barrier in half-space

The solution of (1.4) and (1.5a, c) for a thin barrier in a half-space (I, §6) yields the velocity gradient at the forward stagnation point ($x = 0-$, $y = 0$) in the form

$$q_s' = -(\partial^2 \delta / \partial \xi \partial \eta) \quad (\xi = 0, \eta = -\tfrac{1}{2}\pi), \quad (4.1)$$

where ξ and η are the elliptic co-ordinates of I (6.1), referred to the barrier height, h, as the unit of length, and $\delta(\xi, \eta)$ is given as an expansion in Mathieu functions by the sum of I (6.5) and I (6.7). Substituting these results into (4.1), we obtain (after some reduction)

$$q_s' = (\pi/8)^{\frac{1}{2}}\kappa \sum_{n=0}^{\infty} (-)^n (A_{2n+1})^2 De_1^{2n+1}(g_{e,2n+1}/f_{e,2n+1}) \times \left\{ De_1^{2n+1}(g_{e,2n+1})^2 - (2\kappa/\pi) \sum_{m=1}^{\infty} (b_{2m} - a_{2n+1})^{-1} (B_{2m})^2 De_2^{2m} \right\}, \quad (4.2)$$

where the Mathieu-function parameters on the right-hand side of (4.2) are defined and tabulated, as functions of $s \equiv \kappa^2$, in the National Bureau of Standards (1951) tables. Using these tables, we find that the smallest zero of the right-hand side of (4.2) is $\kappa_s = 2{\cdot}05$. [An independent calculation, based on the variational approximation of I, §4 yields $\kappa_s = 2{\cdot}02$.] This compares with $\kappa_c = 1{\cdot}73$ (see I) for the first appearance of reversed flow downstream of the barrier. We also note that the maximum values of the stagnation-point reversal parameter for the barrier in a channel (§2) are $\kappa_0 = 2{\cdot}08$ ($h/H \uparrow 1$) and $\kappa_1 = 1{\cdot}83$ ($h/H = 0{\cdot}42$).

The solution of (1.4) and (1.5*a*, *c*) for a semi-circular barrier in a half-space (Miles 1968*b*, hereinafter referred to as II) yields

$$q_s' = 1 + (\partial^2\delta/\partial r\,\partial\theta) \quad (r = 1, \theta = \pi), \quad (4.3)$$

where r and θ are polar co-ordinates, referred to the barrier height, $h \equiv a$, as the unit of length, and $y_0 = \sin\theta$ in (1.5*a*). Retaining only the first two of the lee-wave modes in II (3.2), we obtain

$$q_s' = (Y_1Y_2 + b^2J_1J_2)^{-1}\{\kappa(Y_2^2 + b^2J_2^2) - \tfrac{3}{2}b^2\}, \quad (4.4)$$

where $b = 8/(3\pi)$, and $J_{1,2}$ and $Y_{1,2}$ are Bessel functions of argument κ. The smallest zero of the right-hand side of (4.4) is $\kappa_s = 1{\cdot}8$. This compares with $\kappa_c = 1{\cdot}3$ (II, appendix) for the first appearance of reversed flow downstream of the barrier.

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